7.4 Arc Length and Surfaces of Revolution

- **Find the arc length of a smooth curve.**
- **Find the area of a surface of revolution.**

CHRISTIAN HUYGENS (1629–1695)

The Dutch mathematician Christian Huygens, who invented the pendulum clock, and James Gregory (1638–1675), a Scottish mathematician, both made early contributions to the problem of finding the length of a rectifiable curve.

See LarsonCalculus.com to read more of this biography.

Arc Length

In this section, definite integrals are used to find the arc lengths of curves and the areas of surfaces of revolution. In either case, an arc (a segment of a curve) is approximated by straight line segments whose lengths are given by the familiar Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

A **rectifiable** curve is one that has a finite arc length. You will see that a sufficient condition for the graph of a function f to be rectifiable between $(a, f(a))$ and $(b, f(b))$ is that f' be continuous on $[a, b]$. Such a function is **continuously differentiable** on $[a, b]$, and its graph on the interval $[a, b]$ is a **smooth curve.**

Consider a function $y = f(x)$ that is continuously differentiable on the interval $[a, b]$. You can approximate the graph of f by n line segments whose endpoints are determined by the partition

$$a = x_0 < x_1 < x_2 < \cdot\cdot\cdot < x_n = b$$

as shown in Figure 7.37. By letting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, you can approximate the length of the graph by

$$\begin{aligned} s &\approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i). \end{aligned}$$

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. So, the length of the graph is

$$s = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i).$$

Because $f'(x)$ exists for each x in (x_{i-1}, x_i), the Mean Value Theorem guarantees the existence of c_i in (x_{i-1}, x_i) such that

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f'(c_i)(x_i - x_{i-1}) \\ \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} &= f'(c_i) \\ \frac{\Delta y_i}{\Delta x_i} &= f'(c_i). \end{aligned}$$

Because f' is continuous on $[a, b]$, it follows that $\sqrt{1 + [f'(x)]^2}$ is also continuous (and therefore integrable) on $[a, b]$, which implies that

$$\begin{aligned} s &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,(\Delta x_i) \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \end{aligned}$$

where s is called the **arc length** of f between a and b.

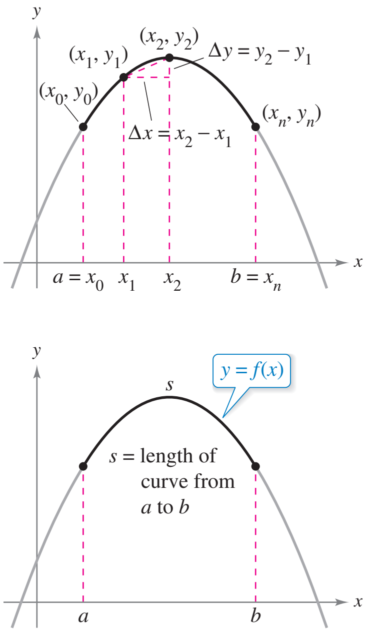


Figure 7.37

Bettmann/Corbis

Definition of Arc Length

Let the function $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, for a smooth curve $x = g(y)$, the **arc length** of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

FOR FURTHER INFORMATION To see how arc length can be used to define trigonometric functions, see the article "Trigonometry Requires Calculus, Not Vice Versa" by Yves Nievergelt in *UMAP Modules.*

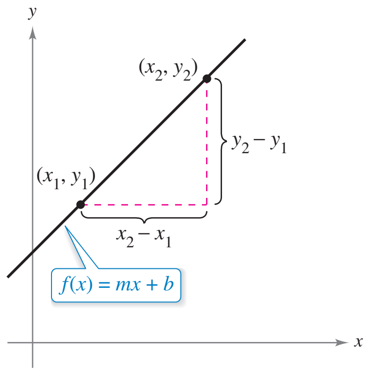
Because the definition of arc length can be applied to a linear function, you can check to see that this new definition agrees with the standard Distance Formula for the length of a line segment. This is shown in Example 1.

EXAMPLE 1 The Length of a Line Segment

Find the arc length from (x_1, y_1) to (x_2, y_2) on the graph of

$$f(x) = mx + b$$

as shown in Figure 7.38.



The formula for the arc length of the graph of f from (x_1, y_1) to (x_2, y_2) is the same as the standard Distance Formula.
Figure 7.38

Solution Because

$$m = f'(x) = \frac{y_2 - y_1}{x_2 - x_1}$$

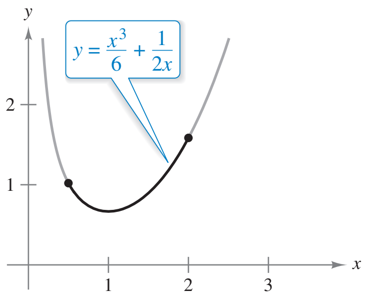
it follows that

$$\begin{aligned}
s &= \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\, dx && \text{Formula for arc length} \\
&= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2}\, dx \\
&= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x)\Big]_{x_1}^{x_2} && \text{Integrate and simplify.} \\
&= \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}\,(x_2 - x_1) \\
&= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\end{aligned}$$

which is the formula for the distance between two points in the plane.

TECHNOLOGY Definite integrals representing arc length often are very difficult to evaluate. In this section, a few examples are presented. In the next chapter, with more advanced integration techniques, you will be able to tackle more difficult arc length problems. In the meantime, remember that you can always use a numerical integration program to approximate an arc length. For instance, use the numerical integration feature of a graphing utility to approximate arc lengths in Examples 2 and 3.

EXAMPLE 2 Finding Arc Length



The arc length of the graph of y on $\left[\frac{1}{2}, 2\right]$
Figure 7.39

Find the arc length of the graph of

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

on the interval $\left[\frac{1}{2}, 2\right]$, as shown in Figure 7.39.

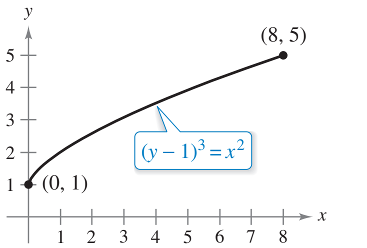
Solution Using

$$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

yields an arc length of

$$\begin{aligned}
s &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx && \text{Formula for arc length} \\
&= \int_{1/2}^2 \sqrt{1 + \left[\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right]^2}\, dx \\
&= \int_{1/2}^2 \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\, dx \\
&= \int_{1/2}^2 \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx && \text{Simplify.} \\
&= \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^2 && \text{Integrate.} \\
&= \frac{1}{2}\left(\frac{13}{6} + \frac{47}{24}\right) \\
&= \frac{33}{16}.
\end{aligned}$$

EXAMPLE 3 Finding Arc Length



The arc length of the graph of y on $[0, 8]$
Figure 7.40

Find the arc length of the graph of $(y - 1)^3 = x^2$ on the interval $[0, 8]$, as shown in Figure 7.40.

Solution Begin by solving for x in terms of y: $x = \pm(y - 1)^{3/2}$. Choosing the positive value of x produces

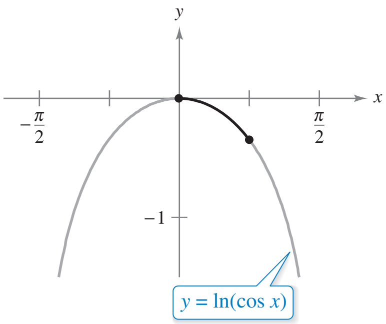
$$\frac{dx}{dy} = \frac{3}{2}(y - 1)^{1/2}.$$

The x-interval $[0, 8]$ corresponds to the y-interval $[1, 5]$, and the arc length is

$$\begin{aligned}
s &= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy && \text{Formula for arc length} \\
&= \int_1^5 \sqrt{1 + \left[\frac{3}{2}(y - 1)^{1/2}\right]^2}\, dy \\
&= \int_1^5 \sqrt{\frac{9}{4}y - \frac{5}{4}}\, dy \\
&= \frac{1}{2}\int_1^5 \sqrt{9y - 5}\, dy && \text{Simplify.} \\
&= \frac{1}{18}\left[\frac{(9y - 5)^{3/2}}{3/2}\right]_1^5 && \text{Integrate.} \\
&= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right) \\
&\approx 9.073.
\end{aligned}$$

EXAMPLE 4 Finding Arc Length

See LarsonCalculus.com for an interactive version of this type of example.



The arc length of the graph of y on $\left[0, \frac{\pi}{4}\right]$

Figure 7.41

Find the arc length of the graph of

$$y = \ln(\cos x)$$

from $x = 0$ to $x = \pi/4$, as shown in Figure 7.41.

Solution Using

$$\frac{dy}{dx} = -\frac{\sin x}{\cos x} = -\tan x$$

yields an arc length of

$$\begin{aligned}
s &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx && \text{Formula for arc length} \\
&= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx \\
&= \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx && \text{Trigonometric identity} \\
&= \int_0^{\pi/4} \sec x\,dx && \text{Simplify.} \\
&= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4} && \text{Integrate.} \\
&= \ln\left(\sqrt{2} + 1\right) - \ln 1 \\
&\approx 0.881.
\end{aligned}$$

EXAMPLE 5 Length of a Cable

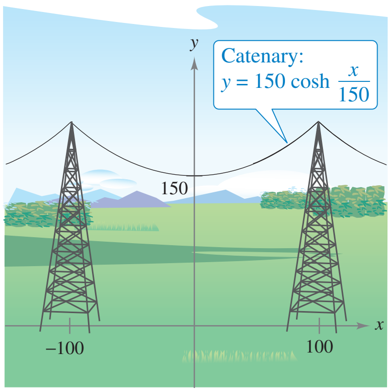


Figure 7.42

An electric cable is hung between two towers that are 200 feet apart, as shown in Figure 7.42. The cable takes the shape of a catenary whose equation is

$$y = 75(e^{x/150} + e^{-x/150}) = 150 \cosh \frac{x}{150}.$$

Find the arc length of the cable between the two towers.

Solution Because $y' = \frac{1}{2}(e^{x/150} - e^{-x/150})$, you can write

$$(y')^2 = \frac{1}{4}(e^{x/75} - 2 + e^{-x/75})$$

and

$$1 + (y')^2 = \frac{1}{4}(e^{x/75} + 2 + e^{-x/75}) = \left[\frac{1}{2}(e^{x/150} + e^{-x/150})\right]^2.$$

Therefore, the arc length of the cable is

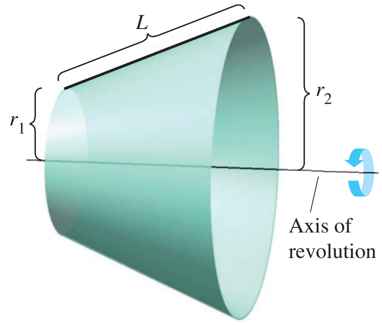
$$\begin{aligned}
s &= \int_a^b \sqrt{1 + (y')^2}\,dx && \text{Formula for arc length} \\
&= \frac{1}{2}\int_{-100}^{100} (e^{x/150} + e^{-x/150})\,dx \\
&= 75\Big[e^{x/150} - e^{-x/150}\Big]_{-100}^{100} && \text{Integrate.} \\
&= 150(e^{2/3} - e^{-2/3}) \\
&\approx 215 \text{ feet.}
\end{aligned}$$

Area of a Surface of Revolution

In Sections 7.2 and 7.3, integration was used to calculate the volume of a solid of revolution. You will now look at a procedure for finding the area of a surface of revolution.

> **Definition of Surface of Revolution**
>
> When the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution.**

The area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone. Consider the line segment in the figure at the right, where L is the length of the line segment, r_1 is the radius at the left end of the line segment, and r_2 is the radius at the right end of the line segment. When the line segment is revolved about its axis of revolution, it forms a frustum of a right circular cone, with



$$S = 2\pi r L \qquad \text{Lateral surface area of frustum}$$

where

$$r = \frac{1}{2}(r_1 + r_2). \qquad \text{Average radius of frustum}$$

(In Exercise 54, you are asked to verify the formula for S.)

Consider a function f that has a continuous derivative on the interval $[a, b]$. The graph of f is revolved about the x-axis to form a surface of revolution, as shown in Figure 7.43. Let Δ be a partition of $[a, b]$, with subintervals of width Δx_i. Then the line segment of length

$$\Delta L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

generates a frustum of a cone. Let r_i be the average radius of this frustum. By the Intermediate Value Theorem, a point d_i exists (in the ith subinterval) such that

$$r_i = f(d_i).$$

The lateral surface area ΔS_i of the frustum is

$$\begin{aligned} \Delta S_i &= 2\pi r_i \, \Delta L_i \\ &= 2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} \\ &= 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i. \end{aligned}$$

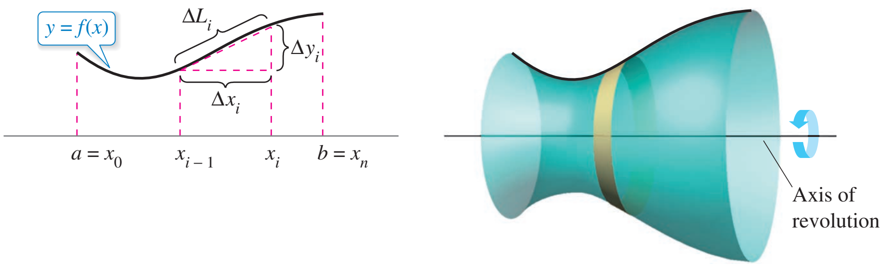


Figure 7.43

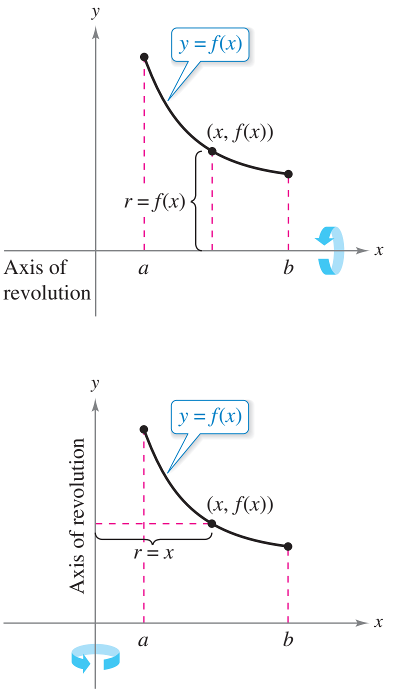


Figure 7.44

By the Mean Value Theorem, a point c_i exists in (x_{i-1}, x_i) such that

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i}.$$

So, $\Delta S_i = 2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x_i$, and the total surface area can be approximated by

$$S \approx 2\pi \sum_{i=1}^{n} f(d_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x_i.$$

It can be shown that the limit of the right side as $\|\Delta\| \to 0$ $(n \to \infty)$ is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

In a similar manner, if the graph of f is revolved about the y-axis, then S is

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\,dx.$$

In these two formulas for S, you can regard the products $2\pi f(x)$ and $2\pi x$ as the circumferences of the circles traced by a point (x, y) on the graph of f as it is revolved about the x-axis and the y-axis (Figure 7.44). In one case, the radius is $r = f(x)$, and in the other case, the radius is $r = x$. Moreover, by appropriately adjusting r, you can generalize the formula for surface area to cover *any* horizontal or vertical axis of revolution, as indicated in the next definition.

Definition of the Area of a Surface of Revolution

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx \qquad y \text{ is a function of } x.$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\,dy \qquad x \text{ is a function of } y.$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

The formulas in this definition are sometimes written as

$$S = 2\pi \int_a^b r(x)\,ds \qquad y \text{ is a function of } x.$$

and

$$S = 2\pi \int_c^d r(y)\,ds \qquad x \text{ is a function of } y.$$

where

$$ds = \sqrt{1 + [f'(x)]^2}\,dx \quad \text{and} \quad ds = \sqrt{1 + [g'(y)]^2}\,dy,$$

respectively.

EXAMPLE 6 The Area of a Surface of Revolution

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0, 1]$ about the x-axis, as shown in Figure 7.45.

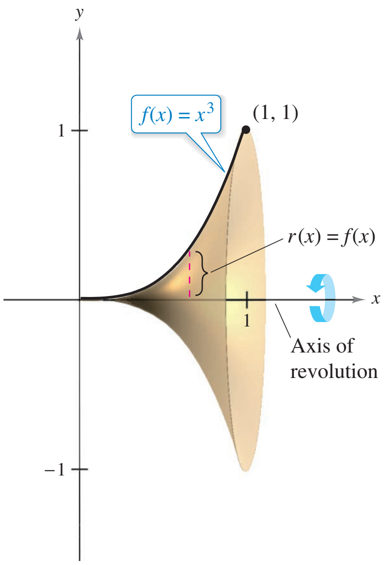


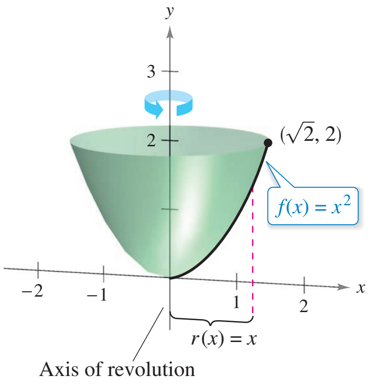
Figure 7.45

Solution The distance between the x-axis and the graph of f is $r(x) = f(x)$, and because $f'(x) = 3x^2$, the surface area is

$$
\begin{aligned}
S &= 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx && \text{Formula for surface area} \\
&= 2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2}\, dx \\
&= \frac{2\pi}{36} \int_0^1 (36x^3)(1 + 9x^4)^{1/2}\, dx && \text{Simplify.} \\
&= \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]_0^1 && \text{Integrate.} \\
&= \frac{\pi}{27}(10^{3/2} - 1) \\
&\approx 3.563.
\end{aligned}
$$

EXAMPLE 7 The Area of a Surface of Revolution

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis, as shown in the figure below.



Solution In this case, the distance between the graph of f and the y-axis is $r(x) = x$. Using $f'(x) = 2x$ and the formula for surface area, you can determine that

$$
\begin{aligned}
S &= 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx \\
&= 2\pi \int_0^{\sqrt{2}} x\sqrt{1 + (2x)^2}\, dx \\
&= \frac{2\pi}{8} \int_0^{\sqrt{2}} (1 + 4x^2)^{1/2}(8x)\, dx && \text{Simplify.} \\
&= \frac{\pi}{4}\left[\frac{(1 + 4x^2)^{3/2}}{3/2}\right]_0^{\sqrt{2}} && \text{Integrate.} \\
&= \frac{\pi}{6}[(1 + 8)^{3/2} - 1] \\
&= \frac{13\pi}{3} \\
&\approx 13.614.
\end{aligned}
$$

7.4 Exercises

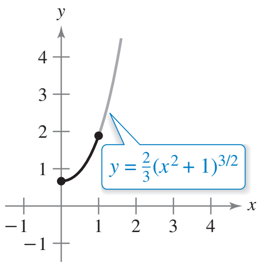
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Distance Using Two Methods **In Exercises 1 and 2, find the distance between the points using (a) the Distance Formula and (b) integration.**

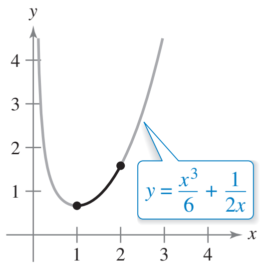
1. $(0, 0),\quad (8, 15)$ **2.** $(1, 2),\quad (7, 10)$

Finding Arc Length **In Exercises 3–16, find the arc length of the graph of the function over the indicated interval.**

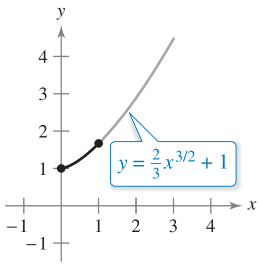
3. $y = \frac{2}{3}(x^2 + 1)^{3/2}$



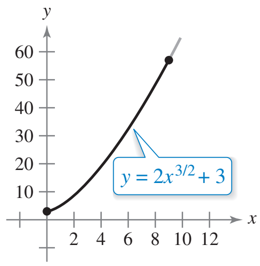
4. $y = \frac{x^3}{6} + \frac{1}{2x}$



5. $y = \frac{2}{3}x^{3/2} + 1$



6. $y = 2x^{3/2} + 3$



7. $y = \frac{3}{2}x^{2/3},\quad [1, 8]$ **8.** $y = \frac{x^4}{8} + \frac{1}{4x^2},\quad [1, 3]$

9. $y = \frac{x^5}{10} + \frac{1}{6x^3},\quad [2, 5]$ **10.** $y = \frac{3}{2}x^{2/3} + 4,\quad [1, 27]$

11. $y = \ln(\sin x),\quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ **12.** $y = \ln(\cos x),\quad \left[0, \frac{\pi}{3}\right]$

13. $y = \frac{1}{2}(e^x + e^{-x}),\quad [0, 2]$

14. $y = \ln\left(\frac{e^x + 1}{e^x - 1}\right),\quad [\ln 2, \ln 3]$

15. $x = \frac{1}{3}(y^2 + 2)^{3/2},\quad 0 \le y \le 4$

16. $x = \frac{1}{3}\sqrt{y}(y - 3),\quad 1 \le y \le 4$

Finding Arc Length **In Exercises 17–26, (a) sketch the graph of the function, highlighting the part indicated by the given interval, (b) find a definite integral that represents the arc length of the curve over the indicated interval and observe that the integral cannot be evaluated with the techniques studied so far, and (c) use the integration capabilities of a graphing utility to approximate the arc length.**

17. $y = 4 - x^2,\quad 0 \le x \le 2$

18. $y = x^2 + x - 2,\quad -2 \le x \le 1$

19. $y = \frac{1}{x},\quad 1 \le x \le 3$

20. $y = \frac{1}{x + 1},\quad 0 \le x \le 1$

21. $y = \sin x,\quad 0 \le x \le \pi$

22. $y = \cos x,\quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

23. $x = e^{-y},\quad 0 \le y \le 2$

24. $y = \ln x,\quad 1 \le x \le 5$

25. $y = 2 \arctan x,\quad 0 \le x \le 1$

26. $x = \sqrt{36 - y^2},\quad 0 \le y \le 3$

Approximation **In Exercises 27 and 28, determine which value best approximates the length of the arc represented by the integral. (Make your selection on the basis of a sketch of the arc, *not* by performing any calculations.)**

27. $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\, dx$

(a) 25 (b) 5 (c) 2 (d) -4 (e) 3

28. $\displaystyle\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\, dx$

(a) 3 (b) -2 (c) 4 (d) $\frac{4\pi}{3}$ (e) 1

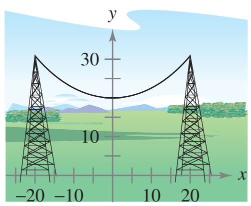
Approximation **In Exercises 29 and 30, approximate the arc length of the graph of the function over the interval [0, 4] in four ways. (a) Use the Distance Formula to find the distance between the endpoints of the arc. (b) Use the Distance Formula to find the lengths of the four line segments connecting the points on the arc when $x = 0$, $x = 1$, $x = 2$, $x = 3$, and $x = 4$. Find the sum of the four lengths. (c) Use Simpson's Rule with $n = 10$ to approximate the integral yielding the indicated arc length. (d) Use the integration capabilities of a graphing utility to approximate the integral yielding the indicated arc length.**

29. $f(x) = x^3$ **30.** $f(x) = (x^2 - 4)^2$

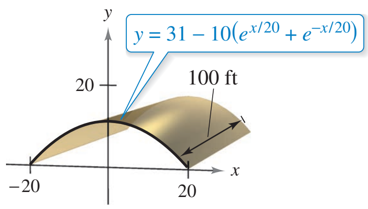
31. Length of a Catenary Electrical wires suspended between two towers form a catenary (see figure) modeled by the equation

$$y = 20 \cosh \frac{x}{20},\quad -20 \le x \le 20$$

where x and y are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.



32. Roof Area A barn is 100 feet long and 40 feet wide (see figure). A cross section of the roof is the inverted catenary $y = 31 - 10(e^{x/20} + e^{-x/20})$. Find the number of square feet of roofing on the barn.



33. Length of Gateway Arch The Gateway Arch in St. Louis, Missouri, is modeled by

$$y = 693.8597 - 68.7672 \cosh 0.0100333x,$$
$$-299.2239 \le x \le 299.2239.$$

(See Section 5.8, Section Project: St. Louis Arch.) Use the integration capabilities of a graphing utility to approximate the length of this curve (see figure).

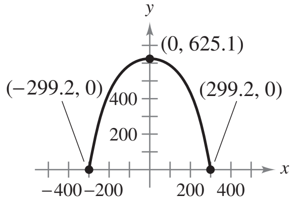


Figure for 33

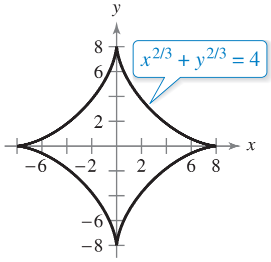


Figure for 34

34. Astroid Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$.

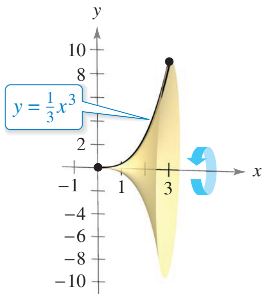
35. Arc Length of a Sector of a Circle Find the arc length from $(0, 3)$ clockwise to $(2, \sqrt{5})$ along the circle $x^2 + y^2 = 9$.

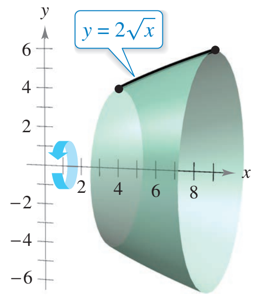
36. Arc Length of a Sector of a Circle Find the arc length from $(-3, 4)$ clockwise to $(4, 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

Finding the Area of a Surface of Revolution In Exercises 37–42, set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the *x*-axis.

37. $y = \frac{1}{3}x^3$

38. $y = 2\sqrt{x}$





39. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}, \quad 1 \le x \le 2$

40. $y = 3x, \quad 0 \le x \le 3$

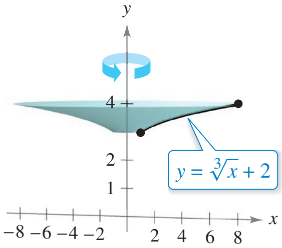
41. $y = \sqrt{4 - x^2}, \quad -1 \le x \le 1$

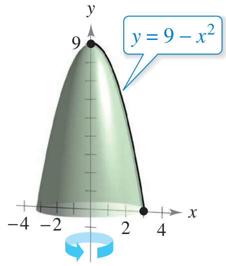
42. $y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

Finding the Area of a Surface of Revolution In Exercises 43–46, set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the *y*-axis.

43. $y = \sqrt[3]{x} + 2$

44. $y = 9 - x^2$





45. $y = 1 - \dfrac{x^2}{4}, \quad 0 \le x \le 2$

46. $y = \dfrac{x}{2} + 3, \quad 1 \le x \le 5$

Finding the Area of a Surface of Revolution In Exercises 47 and 48, use the integration capabilities of a graphing utility to approximate the surface area of the solid of revolution.

	Function	Interval	Axis of Revolution
47.	$y = \sin x$	$[0, \pi]$	x-axis
48.	$y = \ln x$	$[1, e]$	y-axis

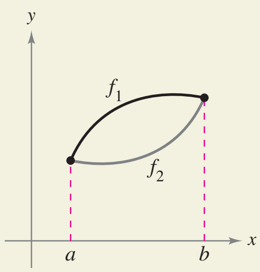
WRITING ABOUT CONCEPTS

49. Rectifiable Curve Define a rectifiable curve.

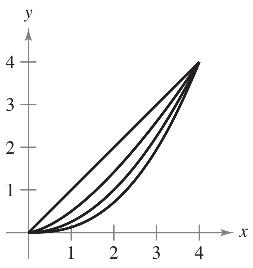
50. Precalculus and Calculus What precalculus formula and representative element are used to develop the integration formula for arc length?

51. Precalculus and Calculus What precalculus formula and representative element are used to develop the integration formula for the area of a surface of revolution?

52. HOW DO YOU SEE IT? The graphs of the functions f_1 and f_2 on the interval $[a, b]$ are shown in the figure. The graph of each function is revolved about the x-axis. Which surface of revolution has the greater surface area? Explain.



53. Think About It The figure shows the graphs of the functions $y_1 = x$, $y_2 = \frac{1}{2}x^{3/2}$, $y_3 = \frac{1}{4}x^2$, and $y_4 = \frac{1}{8}x^{5/2}$ on the interval $[0, 4]$. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(a) Label the functions.

(b) List the functions in order of increasing arc length.

(c) Verify your answer in part (b) by using the integration capabilities of a graphing utility to approximate each arc length accurate to three decimal places.

54. Verifying a Formula

(a) Given a circular sector with radius L and central angle θ (see figure), show that the area of the sector is given by

$$S = \frac{1}{2}L^2\theta.$$

(b) By joining the straight-line edges of the sector in part (a), a right circular cone is formed (see figure) and the lateral surface area of the cone is the same as the area of the sector. Show that the area is $S = \pi rL$, where r is the radius of the base of the cone. (*Hint:* The arc length of the sector equals the circumference of the base of the cone.)

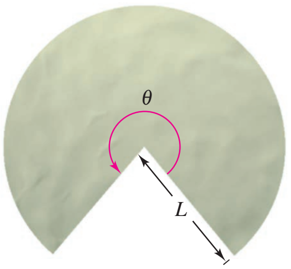


Figure for 54(a)

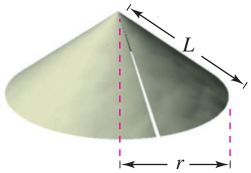
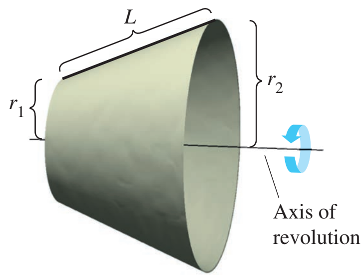


Figure for 54(b)

(c) Use the result of part (b) to verify that the formula for the lateral surface area of the frustum of a cone with slant height L and radii r_1 and r_2 (see figure) is $S = \pi(r_1 + r_2)L$. (*Note:* This formula was used to develop the integral for finding the surface area of a surface of revolution.)



55. Lateral Surface Area of a Cone A right circular cone is generated by revolving the region bounded by $y = 3x/4$, $y = 3$, and $x = 0$ about the y-axis. Find the lateral surface area of the cone.

56. Lateral Surface Area of a Cone A right circular cone is generated by revolving the region bounded by $y = hx/r$, $y = h$, and $x = 0$ about the y-axis. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2 + h^2}$.

57. Using a Sphere Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9 - x^2}$, $0 \le x \le 2$, about the y-axis.

58. Using a Sphere Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}$, $0 \le x \le a$, about the y-axis. Assume that $a < r$.

59. Modeling Data The circumference C (in inches) of a vase is measured at three-inch intervals starting at its base. The measurements are shown in the table, where y is the vertical distance in inches from the base.

y	0	3	6	9	12	15	18
C	50	65.5	70	66	58	51	48

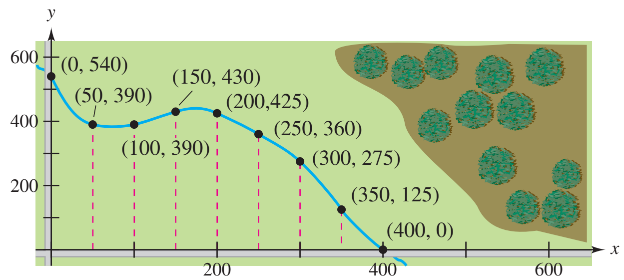
(a) Use the data to approximate the volume of the vase by summing the volumes of approximating disks.

(b) Use the data to approximate the outside surface area (excluding the base) of the vase by summing the outside surface areas of approximating frustums of right circular cones.

(c) Use the regression capabilities of a graphing utility to find a cubic model for the points (y, r), where $r = C/(2\pi)$. Use the graphing utility to plot the points and graph the model.

(d) Use the model in part (c) and the integration capabilities of a graphing utility to approximate the volume and outside surface area of the vase. Compare the results with your answers in parts (a) and (b).

60. Modeling Data Property bounded by two perpendicular roads and a stream is shown in the figure. All distances are measured in feet.



(a) Use the regression capabilities of a graphing utility to fit a fourth-degree polynomial to the path of the stream.

(b) Use the model in part (a) to approximate the area of the property in acres.

(c) Use the integration capabilities of a graphing utility to find the length of the stream that bounds the property.

61. Volume and Surface Area Let R be the region bounded by $y = 1/x$, the x-axis, $x = 1$, and $x = b$, where $b > 1$. Let D be the solid formed when R is revolved about the x-axis.

(a) Find the volume V of D.

(b) Write the surface area S as an integral.

(c) Show that V approaches a finite limit as $b \to \infty$.

(d) Show that $S \to \infty$ as $b \to \infty$.

62. Think About It Consider the equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

(a) Use a graphing utility to graph the equation.

(b) Set up the definite integral for finding the first-quadrant arc length of the graph in part (a).

(c) Compare the interval of integration in part (b) and the domain of the integrand. Is it possible to evaluate the definite integral? Is it possible to use Simpson's Rule to evaluate the definite integral? Explain. (You will learn how to evaluate this type of integral in Section 8.8.)

Approximating Arc Length or Surface Area In Exercises 63–66, set up the definite integral for finding the indicated arc length or surface area. Then use the integration capabilities of a graphing utility to approximate the arc length or surface area. (You will learn how to evaluate this type of integral in Section 8.8.)

63. Length of Pursuit A fleeing object leaves the origin and moves up the y-axis (see figure). At the same time, a pursuer leaves the point $(1, 0)$ and always moves toward the fleeing object. The pursuer's speed is twice that of the fleeing object. The equation of the path is modeled by

$$y = \frac{1}{3}(x^{3/2} - 3x^{1/2} + 2).$$

How far has the fleeing object traveled when it is caught? Show that the pursuer has traveled twice as far.

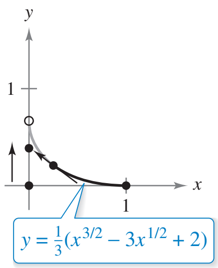


Figure for 63

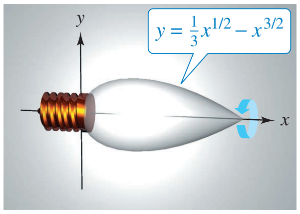


Figure for 64

64. Bulb Design An ornamental light bulb is designed by revolving the graph of

$$y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3},$$

about the x-axis, where x and y are measured in feet (see figure). Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb. (Assume that the glass is 0.015 inch thick.)

65. Astroid Find the area of the surface formed by revolving the portion in the first quadrant of the graph of $x^{2/3} + y^{2/3} = 4$, $0 \le y \le 8$, about the y-axis.

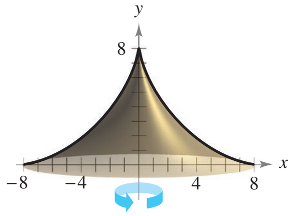


Figure for 65

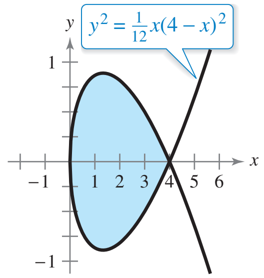


Figure for 66

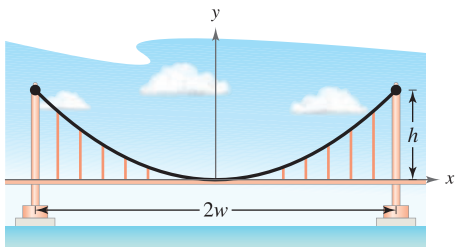
66. Using a Loop Consider the graph of

$$y^2 = \frac{1}{12}x(4 - x)^2$$

shown in the figure. Find the area of the surface formed when the loop of this graph is revolved about the x-axis.

67. Suspension Bridge A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from its lowest point to its highest point and let $2w$ represent the total span of the bridge (see figure). Show that the length C of the cable is given by

$$C = 2\int_0^w \sqrt{1 + (4h^2/w^4)x^2}\, dx.$$



68. Suspension Bridge The Humber Bridge, located in the United Kingdom and opened in 1981, has a main span of about 1400 meters. Each of its towers has a height of about 155 meters. Use these dimensions, the integral in Exercise 67, and the integration capabilities of a graphing utility to approximate the length of a parabolic cable along the main span.

69. Arc Length and Area Let C be the curve given by $f(x) = \cosh x$ for $0 \le x \le t$, where $t > 0$. Show that the arc length of C is equal to the area bounded by C and the x-axis. Identify another curve on the interval $0 \le x \le t$ with this property.

PUTNAM EXAM CHALLENGE

70. Find the length of the curve $y^2 = x^3$ from the origin to the point where the tangent makes an angle of 45° with the x-axis.

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